Lie Symmetries of Yang-Mills Equations

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Abstract

We investigate Lie symmetries of general Yang-Mills equations. For this purpose, we first write down the second prolongation of the symmetry generating vector fields, and compute its action on the Yang-Mills equations. Determining equations are then obtained, and solved completely. Provided that Yang-Mills equations are locally solvable, this allows for a complete classification of their Lie symmetries. Symmetries of Yang-Mills equations in the Lorentz gauge are also investigated. PACS: 02.20.+b

1 Introduction

Consider a system of n-th order nonlinear partial differential equations for a number of independent variables x and dependent variables A:

$$\Delta_\nu(x, A, \partial A, \ldots, \partial^{(n)} A) = 0. \tag{1}$$

By a symmetry of this system, we shall mean any mapping of the independent and dependent variables that transforms an arbitrary solution of (1) into a solution. A Lie symmetry is a symmetry that belongs to a local Lie group of transformations. A Lie symmetry is generated by a differential operator v which is a linear combination of partial derivatives with respect to the x and A. In general, the coefficients of the linear combination depend on x and A.

There is a well-defined method for the determination of all Lie symmetries of Eq. (1). It involves the computation of the so-called n-th prolongation of v, denoted by $\mathrm{pr}^{(n)}v$. The n-th prolongation of v is a linear combination of partial derivatives with respect to x, A, and with respect to all partial derivatives of A up to the n-th order. One can show that, provided Eq. (1) is locally solvable and has maximal rank, v generates a symmetry of (1) if and only if the following holds [1]:

$$\left[\mathrm{pr}^{(n)}v\right] \Delta_\nu = 0 \text{ whenever } \Delta_\nu = 0. \tag{2}$$

This constitutes a system of linear equations for the coefficients of partial derivatives making up the operator v.

In this paper, we shall investigate Lie symmetries of general Yang-Mills equations. Such equations are characterized by local gauge invariance under a compact semisimple Lie group. Their importance can hardly be overestimated, since they form the basis of current theories of the strong and electroweak interactions. Yet, as pointed out in Ref. [2], the techniques of symmetry analysis have not been applied systematically to the Yang-Mills equations. Previous work centered on the SU(2) case, for which Lie symmetries of Yang-Mills equations in the Lorentz gauge, and of so-called self-dual Yang-Mills equations, have been obtained [3, 4].

Just recently, however, Torre investigated what he calls natural symmetries of general Yang-Mills equations [5]. They are generalized symmetries (in the sense of Ref. [1]) that, roughly speaking, have a simple behavior under Poincaré and gauge transformations of the fields. Torre showed that all such symmetries come from the gauge transformations admitted by the equations.

In this paper we shall show that, provided the Yang-Mills equations are locally solvable, their Lie symmetries all come from local gauge transformations and conformal transformations. Our results are thus consistent with Torre's. In one aspect they are less general, since we investigate Lie instead of generalized symmetries. In another aspect, however, they are more general, since we do not assume any specific behavior under Poincaré and gauge transformations. We should also point out that Torre makes use of the spinor formalism, applicable to four-dimensional manifolds. Our method can straightforwardly be adapted to higher-dimensional manifolds.

In Section 2, we write down general Yang-Mills equations, the prolongation formulas, and discuss the question of local solvability. Determining equations associated with Yang-Mills equations are obtained in Section 3, and completely solved in Section 4. Yang-Mills equations in the Lorentz gauge are investigated in Section 5.

2 Yang-Mills Equations

With suitable choice of units, Yang-Mills equations can be written as [6, 7]

$$\partial_\mu F_a^{\mu\nu} + C_{abc} A_{b\mu} F_c^{\mu\nu} = 0, \tag{3}$$

where

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + C_{abc} A_b^\mu A_c^\nu. \tag{4}$$

Here ∂_μ represents a partial derivative with respect to the independent variable $x^\mu (\mu = 0, 1, 2, 3)$. The A_b^μ are dependent variables. Greek indices, associated with space-time, are raised and lowered with the Minkowski metric $g_{\mu\nu}$, with signature $(+1, -1, -1, -1)$. Latin indices are associated with the structure constants C_{abc} of a compact semisimple Lie group. The C_{abc} can always be chosen so that they are completely antisymmetric and satisfy

$$C_{acd} C_{bcd} = \delta_{ab}, \tag{5}$$

where δ_{ab} is the Kronecker delta.

Eliminating F from Eqs. (3) and (4), we find that the Yang-Mills equations can be written as

$$\begin{aligned} &\partial_\mu \partial^\mu A_a^\nu - \partial_\mu \partial^\nu A_a^\mu + 2C_{abc} A_b^\mu \partial_\mu A_c^\nu + C_{abc}(\partial_\mu A_b^\mu) A_c^\nu \\ &\quad - C_{abc} A_{b\mu} \partial^\nu A_c^\mu + C_{abc} C_{cdl} A_d^\mu A_l^\nu A_{b\mu} = 0. \end{aligned} \tag{6}$$

The Yang-Mills equations are second-order nonlinear partial differential equations. Generators of symmetry transformations are given by

$$v = H^\kappa \partial_\kappa + \Phi_d^\kappa \frac{\partial}{\partial A_d^\kappa}, \tag{7}$$

where H^κ and Φ^κ_d are functions of x^μ and A^ν_a. The second prolongation of v is given by

$$\mathrm{pr}^{(2)} v = H^\kappa \partial_\kappa + \Phi^\kappa_d \frac{\partial}{\partial A^\kappa_d} + \Phi^\kappa_{d\lambda} \frac{\partial}{\partial(\partial_\lambda A^\kappa_d)} + \Phi^\kappa_{d(\lambda\pi)} \frac{\partial}{\partial(\partial_\lambda \partial_\pi A^\kappa_d)}. \tag{8}$$

The parentheses around λ and π indicate that the implicit summation is restricted to values of the indices such that $\lambda \leq \pi$, that is, over distinct partial derivatives only. The coefficients $\Phi^\kappa_{d\lambda}$ and $\Phi^\kappa_{d\lambda\pi}$ are functions of x^μ and A^ν_a. They are given by [1]

$$\Phi^\kappa_{d\lambda} = D_\lambda \left\{ \Phi^\kappa_d - H^\mu \partial_\mu A^\kappa_d \right\} + H^\mu \partial_\lambda \partial_\mu A^\kappa_d, \tag{9}$$

$$\Phi^\kappa_{d\lambda\pi} = D_\pi D_\lambda \left\{ \Phi^\kappa_d - H^\mu \partial_\mu A^\kappa_d \right\} + H^\mu \partial_\pi \partial_\lambda \partial_\mu A^\kappa_d. \tag{10}$$

The operator D_λ and D_π are total derivatives, that is,

$$D_\lambda = \partial_\lambda + (\partial_\lambda A^\alpha_n) \frac{\partial}{\partial A^\alpha_n} + (\partial_\lambda \partial_\beta A^\alpha_n) \frac{\partial}{\partial(\partial_\beta A^\alpha_n)} + (\partial_\lambda \partial_{(\beta} \partial_{\gamma)} A^\alpha_n) \frac{\partial}{\partial(\partial_\beta \partial_\gamma A^\alpha_n)}. \tag{11}$$

Note that, in the last term, summation is restricted to values of the indices such that $\beta \leq \gamma$.

In this paper, we shall let Eq. (1) represent the Yang-Mills equations, and investigate the most general functions H^κ and Φ^κ_d that satisfy Eq. (2). As shown in Ref. [1], Eq. (2) is a sufficient condition for v to generate a symmetry. Provided the Yang-Mills equations are locally solvable and have maximal rank, this is also a necessary condition.

It is easy to check that the Yang-Mills equations have maximal rank in the sense of Ref. [1]. It is less easy to prove that they are locally solvable. Local solvability means that one can find solutions for arbitrary values of the partial derivatives at a given point, compatible with the equations. A sufficient (but not necessary) condition for local solvability is that the equations be in (general) Kovalevskaya form. Unfortunately, the Yang-Mills equations are not in that form. Nevertheless, it is likely that the Yang-Mills equations are locally solvable. As pointed out in Ref. [1], the main reason why an analytic system is not locally solvable is the existence of additional constraints on partial derivatives implied by the equations themselves. It might appear that such additional constraints could be put on the Yang-Mills fields by acting on Eq. (6) with the operator ∂_ν. The first two terms then vanish, yielding an equation for the functions A^μ_b and their partial derivatives. But we show, in Appendix A, that the resulting equation holds as an identity. Therefore, that operation gives no additional constraints on the fields.

A word on notations. It has already been said that parentheses enclosing a pair of indices indicate that the implicit summation should be carried out only on distinct pairs of indices. It has also been assumed that the implicit summation convention, on Greek as well as Latin indices, is effective. There will, however, be instances where we will not want to sum over repeated indices. Obviously, we could just put summation signs where needed, and no such signs elsewhere. However, the summation convention is so useful that it is better to proceed otherwise. We shall use the summation convention on repeated indices, *unless* indices have a caret, in which case no summation will be carried out. This means, for instance, that in an equation like

$$M^\mu_\mu = N^{\hat{\alpha}}_{\hat{\alpha}}, \tag{12}$$

summation is carried out over μ but not over $\hat{\alpha}$, the latter index having a specific value.

For later purposes, it is useful to write down the Yang-Mills equations (6) in a form that exhibits each second-order partial derivative. For each value of the index a, there are four equations, corresponding to each value of the index ν. They are given by

$$\begin{aligned} \partial_1 \partial_1 A^0_a = & -\partial_2 \partial_2 A^0_a - \partial_3 \partial_3 A^0_a - \partial_1 \partial_0 A^1_a - \partial_2 \partial_0 A^2_a - \partial_3 \partial_0 A^3_a \\ & + 2 C_{abc} A^\mu_b \partial_\mu A^0_c + C_{abc} (\partial_\mu A^\mu_b) A^0_c - C_{abc} A_{b\mu} \partial^0 A^\mu_c + C_{abc} C_{cdl} A^\mu_d A^0_l A_{b\mu}. \end{aligned} \tag{13}$$

$$\partial_2\partial_2 A_a^1 = -\partial_3\partial_3 A_a^1 + \partial_0\partial_0 A_a^1 + \partial_2\partial_1 A_a^2 + \partial_3\partial_1 A_a^3 + \partial_0\partial_1 A_a^0 \\ + 2C_{abc}A_b^\mu\partial_\mu A_c^1 + C_{abc}(\partial_\mu A_b^\mu)A_c^1 - C_{abc}A_{b\mu}\partial^1 A_c^\mu + C_{abc}C_{cdl}A_d^\mu A_l^1 A_{b\mu}. \quad (14)$$

$$\partial_1\partial_1 A_a^2 = -\partial_3\partial_3 A_a^2 + \partial_0\partial_0 A_a^2 + \partial_1\partial_2 A_a^1 + \partial_3\partial_2 A_a^3 + \partial_0\partial_2 A_a^0 \\ + 2C_{abc}A_b^\mu\partial_\mu A_c^2 + C_{abc}(\partial_\mu A_b^\mu)A_c^2 - C_{abc}A_{b\mu}\partial^2 A_c^\mu + C_{abc}C_{cdl}A_d^\mu A_l^2 A_{b\mu}. \quad (15)$$

$$\partial_1\partial_1 A_a^3 = -\partial_2\partial_2 A_a^3 + \partial_0\partial_0 A_a^3 + \partial_1\partial_3 A_a^1 + \partial_2\partial_3 A_a^2 + \partial_0\partial_3 A_a^0 \\ + 2C_{abc}A_b^\mu\partial_\mu A_c^3 + C_{abc}(\partial_\mu A_b^\mu)A_c^3 - C_{abc}A_{b\mu}\partial^3 A_c^\mu + C_{abc}C_{cdl}A_d^\mu A_l^3 A_{b\mu}. \quad (16)$$

3 Determining Equations

In this section, we will translate condition (2) for the Yang-Mills case in explicit equations. First, we have to compute the coefficients $\Phi^\kappa_{d\lambda}$ and $\Phi^\kappa_{d\lambda\pi}$ that appear in the prolongation formula (8). Substituting Eq. (11) into Eqs. (9) and (10), we find

$$\Phi^\kappa_{d\lambda} = \partial_\lambda\Phi_d^\kappa - (\partial_\lambda H^\mu)\partial_\mu A_d^\kappa + (\partial_\lambda A_n^\alpha)\frac{\partial\Phi_d^\kappa}{\partial A_n^\alpha} - (\partial_\lambda A_n^\alpha)(\partial_\mu A_d^\kappa)\frac{\partial H^\mu}{\partial A_n^\alpha}, \quad (17)$$

$$\begin{aligned}\Phi^\kappa_{d\lambda\pi} = {}& \partial_\pi\partial_\lambda\Phi_d^\kappa - (\partial_\mu A_d^\kappa)\partial_\pi\partial_\lambda H^\mu + (\partial_\lambda A_n^\alpha)\frac{\partial}{\partial A_n^\alpha}\partial_\pi\Phi_d^\kappa + (\partial_\pi A_n^\alpha)\frac{\partial}{\partial A_n^\alpha}\partial_\lambda\Phi_d^\kappa \\ &- (\partial_\lambda A_n^\alpha)(\partial_\mu A_d^\kappa)\frac{\partial}{\partial A_n^\alpha}\partial_\pi H^\mu - (\partial_\pi A_n^\alpha)(\partial_\mu A_d^\kappa)\frac{\partial}{\partial A_n^\alpha}\partial_\lambda H^\mu + (\partial_\pi A_p^\beta)(\partial_\lambda A_n^\alpha)\frac{\partial^2\Phi_d^\kappa}{\partial A_p^\beta\partial A_n^\alpha} \\ &- (\partial_\pi\partial_\mu A_d^\kappa)\partial_\lambda H^\mu - (\partial_\lambda\partial_\mu A_d^\kappa)\partial_\pi H^\mu + (\partial_\pi\partial_\lambda A_n^\alpha)\frac{\partial\Phi_d^\kappa}{\partial A_n^\alpha} \\ &- (\partial_\pi A_p^\beta)(\partial_\lambda A_n^\alpha)(\partial_\mu A_d^\kappa)\frac{\partial^2 H^\mu}{\partial A_p^\beta\partial A_n^\alpha} - (\partial_\pi A_n^\alpha)(\partial_\lambda\partial_\mu A_d^\kappa)\frac{\partial H^\mu}{\partial A_n^\alpha} \\ &- (\partial_\lambda A_n^\alpha)(\partial_\pi\partial_\mu A_d^\kappa)\frac{\partial H^\mu}{\partial A_n^\alpha} - (\partial_\mu A_d^\kappa)(\partial_\pi\partial_\lambda A_n^\alpha)\frac{\partial H^\mu}{\partial A_n^\alpha}. \quad (18)\end{aligned}$$

We note that restrictions on summations have disappeared. Applying the prolongation operator (8) to Eq. (6), we obtain

$$\Phi_d^\kappa C_{adc}(2\partial_\kappa A_c^\nu - \partial^\nu A_{c\kappa}) + \Phi_d^\nu C_{abd}\partial_\mu A_b^\mu + \Phi_d^\kappa(C_{abc}C_{cdl} + C_{adc}C_{cbl})A_l^\nu A_{b\kappa} \\ + \Phi_d^\nu C_{abc}C_{cld}A_l^\mu A_{b\mu} + \Phi^\nu_{d\lambda}2C_{abd}A_b^\lambda + \Phi^\kappa_{d\kappa}C_{adc}A_c^\nu - \Phi_d^{\kappa\nu}C_{abd}A_{b\kappa} + \Phi^{\nu\ \ \lambda}_{a\lambda} - \Phi^{\lambda\ \ \nu}_{a\lambda} = 0. \quad (19)$$

Again, restrictions on summations have disappeared.

We now substitute Eqs. (17) and (18) into (19). Regrouping coefficients of various derivatives of A, we get

$$\Big\{\partial_\lambda\partial^\lambda\Phi_a^\nu - \partial_\lambda\partial^\nu\Phi_a^\lambda + A_b^\lambda\left[2C_{abd}\partial_\lambda\Phi_d^\nu - g_\lambda^\nu C_{abd}\partial_\kappa\Phi_d^\kappa - C_{abd}\partial^\nu\Phi_{d\lambda}\right] \\ + A_l^\mu A_{b\kappa}\left[g_\mu^\nu(C_{abc}C_{cdl} + C_{adc}C_{cbl})\Phi_d^\kappa + g_\mu^\kappa C_{abc}C_{cld}\Phi_d^\nu\right]\Big\} \\ + (\partial_\lambda A_n^\alpha)\Big\{2g_\alpha^\nu C_{adn}\Phi_d^\lambda + g^{\lambda\nu}C_{and}\Phi_{d\alpha} + g_\alpha^\lambda C_{and}\Phi_d^\nu - \delta_{an}g_\alpha^\nu\partial_\mu\partial^\mu H^\lambda + \delta_{an}\partial_\alpha\partial^\nu H^\lambda \\ + 2\frac{\partial}{\partial A_n^\alpha}\partial^\lambda\Phi_a^\nu - \frac{\partial}{\partial A_n^\alpha}\partial^\nu\Phi_a^\lambda - g^{\nu\lambda}\frac{\partial}{\partial A_n^\alpha}\partial_\mu\Phi_a^\mu$$

$$
\begin{aligned}
&+ A_b^\mu \Big[-2\delta_{nd} g_\alpha^\nu C_{abd} \partial_\mu H^\lambda + g_\mu^\nu C_{abn} \partial_\alpha H^\lambda + g_{\mu\alpha} C_{abn} \partial^\nu H^\lambda \\
&+\ 2 g_\mu^\lambda C_{abd} \frac{\partial \Phi_d^\nu}{\partial A_n^\alpha} - g_\mu^\nu C_{abd} \frac{\partial \Phi_d^\lambda}{\partial A_n^\alpha} - g^{\nu\lambda} C_{abd} \frac{\partial \Phi_{d\mu}}{\partial A_n^\alpha} \Big] \Big\} \\
&+ (\partial_\kappa A_n^\alpha)(\partial_\lambda A_p^\beta) \Big\{ g^{\kappa\lambda} \frac{\partial^2 \Phi_a^\nu}{\partial A_p^\beta \partial A_n^\alpha} - g^{\nu\lambda} \frac{\partial^2 \Phi_a^\kappa}{\partial A_p^\beta \partial A_n^\alpha} \\
&+ \delta_{an} g_\alpha^\lambda \frac{\partial}{\partial A_p^\beta} \partial^\nu H^\kappa - 2\delta_{an} g_\alpha^\nu \frac{\partial}{\partial A_p^\beta} \partial^\lambda H^\kappa + \delta_{ap} g^{\nu\kappa} \frac{\partial}{\partial A_n^\alpha} \partial_\beta H^\lambda \\
&+\ A_b^\mu \left[-2 g_\mu^\kappa g_\beta^\nu C_{abp} \frac{\partial H^\lambda}{\partial A_n^\alpha} + g_\mu^\nu g_\beta^\kappa C_{abp} \frac{\partial H^\lambda}{\partial A_n^\alpha} + g_{\mu\beta} g^{\nu\kappa} C_{abp} \frac{\partial H^\lambda}{\partial A_n^\alpha} \right] \Big\} \\
&+ (\partial_\kappa A_n^\alpha)(\partial_\lambda A_p^\beta)(\partial_\mu A_a^\pi) \left\{ g^{\nu\lambda} g_\pi^\kappa \frac{\partial^2 H^\mu}{\partial A_p^\beta \partial A_n^\alpha} - g^{\kappa\lambda} g_\pi^\nu \frac{\partial^2 H^\mu}{\partial A_p^\beta \partial A_n^\alpha} \right\} \\
&+ (\partial_\lambda \partial_\mu A_n^\alpha) \left\{ -2\delta_{an} g_\alpha^\nu \partial^\lambda H^\mu + \delta_{an} g^{\lambda\nu} \partial_\alpha H^\mu + \delta_{an} g_\alpha^\lambda \partial^\nu H^\mu + g^{\lambda\mu} \frac{\partial \Phi_a^\nu}{\partial A_n^\alpha} - g^{\nu\mu} \frac{\partial \Phi_a^\lambda}{\partial A_n^\alpha} \right\} \\
&+ (\partial^\kappa A_n^\alpha)(\partial_\lambda \partial_\mu A_p^\beta) \Big\{ -2\delta_{ap} g_\kappa^\lambda g_\beta^\nu \frac{\partial H^\mu}{\partial A_n^\alpha} + \delta_{ap} g_\kappa^\nu g_\beta^\lambda \frac{\partial H^\mu}{\partial A_n^\alpha} + \delta_{ap} g_{\kappa\beta} g^{\lambda\nu} \frac{\partial H^\mu}{\partial A_n^\alpha} \\
&- \delta_{an} g^{\lambda\mu} g_\alpha^\nu \frac{\partial H_\kappa}{\partial A_p^\beta} + \delta_{an} g^{\lambda\nu} g_\alpha^\mu \frac{\partial H_\kappa}{\partial A_p^\beta} \Big\} = 0. \qquad (20)
\end{aligned}
$$

Eq. (2) means that Eq. (20) should hold whenever the Yang-Mills equations hold. To investigate this requirement, we must substitute Eqs. (13)–(16) into (20), and see under what conditions the resulting equations vanish identically. In other words, we have to investigate the conditions under which the coefficients of independent combinations of derivatives of A vanish. This is what we proceed to do.

$\partial A \partial\partial A$ terms

Eqs. (13)–(16) do not involve terms $\partial_\lambda \partial_\mu A_p^\beta$ with λ, μ and β all different. In Eq. (20), therefore, the coefficient of each term $(\partial^\kappa A_n^\alpha)(\partial_\lambda \partial_\mu A_p^\beta)$, with λ, μ and β all different, must vanish. Since $\partial_\lambda \partial_\mu = \partial_\mu \partial_\lambda$, the coefficient must be symmetrized in λ and μ. So $\forall n, p, a, \kappa, \alpha, \nu$ and $\forall \lambda, \mu, \beta \neq$, we must have

$$
\begin{aligned}
&-2\delta_{ap} g_\kappa^\lambda g_\beta^\nu \frac{\partial H^\mu}{\partial A_n^\alpha} - 2\delta_{ap} g_\kappa^\mu g_\beta^\nu \frac{\partial H^\lambda}{\partial A_n^\alpha} + \delta_{ap} g_{\kappa\beta} g^{\lambda\nu} \frac{\partial H^\mu}{\partial A_n^\alpha} \\
&\qquad + \delta_{ap} g_{\kappa\beta} g^{\mu\nu} \frac{\partial H^\lambda}{\partial A_n^\alpha} + \delta_{an} g^{\lambda\nu} g_\alpha^\mu \frac{\partial H_\kappa}{\partial A_p^\beta} + \delta_{an} g^{\mu\nu} g_\alpha^\lambda \frac{\partial H_\kappa}{\partial A_p^\beta} = 0. \qquad (21)
\end{aligned}
$$

Letting $n = a \neq p$ yields, $\forall p, \kappa, \alpha, \nu$ and $\forall \lambda, \mu, \beta \neq$

$$
(g^{\lambda\nu} g_\alpha^\mu + g^{\mu\nu} g_\alpha^\lambda) \frac{\partial H_\kappa}{\partial A_p^\beta} = 0. \qquad (22)
$$

We can set $\nu = \lambda \neq \mu = \alpha$ and obtain $\forall p, \kappa, \beta$:

$$
\frac{\partial H^\kappa}{\partial A_p^\beta} = 0. \qquad (23)
$$

Eq. (23) is a necessary condition for the $\partial A \partial\partial A$ terms (with λ, μ and ν all different) to vanish. Obviously, it makes all $\partial A \partial\partial A$ terms in (20) vanish. Therefore, there is no need to substitute the Yang-Mills equations in those terms.

$\partial\partial A$ terms

Here again, we begin by looking at terms $\partial_\lambda\partial_\mu A_n^\alpha$ in Eq. (20) with λ, μ and α all different. The (symmetrized) coefficients of these terms must vanish. This means that $\forall n, a, \nu$ and $\forall \lambda, \mu, \alpha \neq$

$$-2\delta_{an} g_\alpha^\nu (\partial^\lambda H^\mu + \partial^\mu H^\lambda) + \delta_{an}(g^{\lambda\nu}\partial_\alpha H^\mu + g^{\mu\nu}\partial_\alpha H^\lambda) - g^{\nu\mu}\frac{\partial \Phi_a^\lambda}{\partial A_n^\alpha} - g^{\nu\lambda}\frac{\partial \Phi_a^\mu}{\partial A_n^\alpha} = 0. \tag{24}$$

Taking $a \neq n$ and $\nu = \mu$, we find that $\forall a, n \neq$ and $\forall \lambda, \alpha \neq$

$$\frac{\partial \Phi_a^\lambda}{\partial A_n^\alpha} = 0. \tag{25}$$

On the other hand, setting $a = n$ in Eq. (24), we get $\forall \hat{a}, \nu$ and $\forall \lambda, \mu, \alpha \neq$

$$-2g_\alpha^\nu (\partial^\lambda H^\mu + \partial^\mu H^\lambda) + g^{\lambda\nu}\left\{\partial_\alpha H^\mu - \frac{\partial \Phi_{\hat{a}}^\mu}{\partial A_{\hat{a}}^\alpha}\right\} + g^{\mu\nu}\left\{\partial_\alpha H^\lambda - \frac{\partial \Phi_{\hat{a}}^\lambda}{\partial A_{\hat{a}}^\alpha}\right\} = 0. \tag{26}$$

Setting $\nu = \alpha$ yields, $\forall \lambda, \mu \neq$

$$\partial^\lambda H^\mu + \partial^\mu H^\lambda = 0, \tag{27}$$

while setting $\nu = \lambda$ yields, $\forall \hat{a}$ and $\forall \mu, \alpha \neq$

$$\partial_\alpha H^\mu - \frac{\partial \Phi_{\hat{a}}^\mu}{\partial A_{\hat{a}}^\alpha} = 0. \tag{28}$$

Eqs. (25), (27) and (28) are necessary and sufficient conditions for $\partial\partial A$ terms with λ, μ and α all different to vanish.

Let us now turn to $\partial\partial A$ terms with $\lambda = \mu = \alpha$. Since the Yang-Mills equations (13)–(16) do not involve such terms, their coefficients can be set equal to zero. This yields, $\forall a, n, \nu, \hat{\alpha}$

$$-2\delta_{an} g_{\hat{\alpha}}^\nu \partial^{\hat{\alpha}} H^{\hat{\alpha}} + \delta_{an} g^{\hat{\alpha}\nu}\partial_{\hat{\alpha}} H^{\hat{\alpha}} + \delta_{an} g_{\hat{\alpha}}^{\hat{\alpha}}\partial^\nu H^{\hat{\alpha}} + g^{\hat{\alpha}\hat{\alpha}}\frac{\partial \Phi_a^\nu}{\partial A_n^{\hat{\alpha}}} - g^{\nu\hat{\alpha}}\frac{\partial \Phi_a^{\hat{\alpha}}}{\partial A_n^{\hat{\alpha}}} = 0. \tag{29}$$

Making use of Eqs. (25), (27) and (28), and considering in turn cases where a is equal to n or not, and where ν is equal to $\hat{\alpha}$ or not, it is not difficult to see that (29) holds identically.

We must now turn to the $\partial\partial A$ terms in Eq. (20) with two and only two of the indices λ, μ, α equal. For each value of n, there are 24 such terms. Since these second-order partial derivatives are constrained by the Yang-Mills equations, their coefficients cannot separately be set equal to zero. We have to use the Yang-Mills equations to eliminate some of the second-order partial derivatives, and set equal to zero the coefficients of the remaining independent ones.

So we substitute Eqs. (13)–(16) in (20), thereby eliminating, for every value of n, the following derivatives: $\partial_1\partial_1 A_n^0$, $\partial_2\partial_2 A_n^1$, $\partial_1\partial_1 A_n^2$, and $\partial_1\partial_1 A_n^3$. The coefficients of the remaining second-order partial derivatives are then extracted, and set equal to zero. After minor cancellations, there result the following equations, which hold $\forall a, n, \nu$ and for $\hat{\mu}$ and $\hat{\alpha}$ as indicated.

Coefficient of $\partial_{\hat{\mu}}\partial_{\hat{\mu}} A_n^0$, with $\hat{\mu} = 2, 3$:

$$\delta_{an}\left\{-2g_0^\nu \partial^{\hat{\mu}} H^{\hat{\mu}} + g^{\hat{\mu}\nu}\partial_0 H^{\hat{\mu}} + 2g_0^\nu \partial^1 H^1 - g^{1\nu}\partial_0 H^1\right\} - g^{\nu\hat{\mu}}\frac{\partial \Phi_a^{\hat{\mu}}}{\partial A_n^0} + g^{\nu 1}\frac{\partial \Phi_a^1}{\partial A_n^0} = 0. \tag{30}$$

Coefficient of $\partial_{\hat{\mu}}\partial_{\hat{\mu}} A_n^1$, with $\hat{\mu} = 0, 3$:

$$\delta_{an}\left\{-2g_1^\nu \partial^{\hat{\mu}} H^{\hat{\mu}} + g^{\hat{\mu}\nu}\partial_1 H^{\hat{\mu}} + g^{\hat{\mu}\hat{\mu}}\left[-2g_1^\nu \partial^2 H^2 + g^{2\nu}\partial_1 H^2\right]\right\} - g^{\nu\hat{\mu}}\frac{\partial \Phi_a^{\hat{\mu}}}{\partial A_n^1} - g^{\hat{\mu}\hat{\mu}} g^{\nu 2}\frac{\partial \Phi_a^2}{\partial A_n^1} = 0. \tag{31}$$

Coefficient of $\partial_{\hat{\mu}}\partial_{\hat{\mu}}A^2_n$, with $\hat{\mu}=0,3$:

$$\delta_{an}\left\{-2g^\nu_2\partial^{\hat{\mu}}H^{\hat{\mu}}+g^{\hat{\mu}\nu}\partial_2H^{\hat{\mu}}+g^{\hat{\mu}\hat{\mu}}\left[-2g^\nu_2\partial^1H^1+g^{1\nu}\partial_2H^1\right]\right\}-g^{\nu\hat{\mu}}\frac{\partial\Phi^{\hat{\mu}}_a}{\partial A^2_n}-g^{\hat{\mu}\hat{\mu}}g^{\nu 1}\frac{\partial\Phi^1_a}{\partial A^2_n}=0. \quad (32)$$

Coefficient of $\partial_{\hat{\mu}}\partial_{\hat{\mu}}A^3_n$, with $\hat{\mu}=0,2$:

$$\delta_{an}\left\{-2g^\nu_3\partial^{\hat{\mu}}H^{\hat{\mu}}+g^{\hat{\mu}\nu}\partial_3H^{\hat{\mu}}+g^{\hat{\mu}\hat{\mu}}\left[-2g^\nu_3\partial^1H^1+g^{1\nu}\partial_3H^1\right]\right\}-g^{\nu\hat{\mu}}\frac{\partial\Phi^{\hat{\mu}}_a}{\partial A^3_n}-g^{\hat{\mu}\hat{\mu}}g^{\nu 1}\frac{\partial\Phi^1_a}{\partial A^3_n}=0. \quad (33)$$

Coefficient of $\partial_{\hat{\alpha}}\partial_0A^{\hat{\alpha}}_n$, with $\hat{\alpha}=1,2,3$:

$$\begin{aligned}&\delta_{an}\left\{-g^\nu_{\hat{\alpha}}(\partial^{\hat{\alpha}}H^0+2\partial^0H^{\hat{\alpha}})+g^{0\nu}(\partial_{\hat{\alpha}}H^{\hat{\alpha}}+2\partial^1H^1)+g^{\hat{\alpha}}_{\hat{\alpha}}\partial^\nu H^0-g^{1\nu}\partial_0H^1\right\}\\&-g^{\nu 0}\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}-g^{\nu\hat{\alpha}}\frac{\partial\Phi^0_a}{\partial A^{\hat{\alpha}}_n}-g^{11}\frac{\partial\Phi^\nu_a}{\partial A^0_n}+g^{\nu 1}\frac{\partial\Phi^1_a}{\partial A^0_n}=0.\end{aligned} \quad (34)$$

Coefficient of $\partial_{\hat{\alpha}}\partial_1A^{\hat{\alpha}}_n$, with $\hat{\alpha}=0,2,3$:

$$\begin{aligned}&\delta_{an}\left\{-g^\nu_{\hat{\alpha}}(\partial^{\hat{\alpha}}H^1+2\partial^1H^{\hat{\alpha}})+g^{1\nu}(\partial_{\hat{\alpha}}H^{\hat{\alpha}}+2\partial^2H^2)+g^{\hat{\alpha}}_{\hat{\alpha}}\partial^\nu H^1+g^{2\nu}\partial_1H^2\right\}\\&-g^{\nu 1}\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}-g^{\nu\hat{\alpha}}\frac{\partial\Phi^1_a}{\partial A^{\hat{\alpha}}_n}+g^{22}\frac{\partial\Phi^\nu_a}{\partial A^1_n}-g^{\nu 2}\frac{\partial\Phi^2_a}{\partial A^1_n}=0.\end{aligned} \quad (35)$$

Coefficient of $\partial_{\hat{\alpha}}\partial_2A^{\hat{\alpha}}_n$, with $\hat{\alpha}=0,1,3$:

$$\begin{aligned}&\delta_{an}\left\{-g^\nu_{\hat{\alpha}}(\partial^{\hat{\alpha}}H^2+2\partial^2H^{\hat{\alpha}})+g^{2\nu}(\partial_{\hat{\alpha}}H^{\hat{\alpha}}+2\partial^1H^1)+g^{\hat{\alpha}}_{\hat{\alpha}}\partial^\nu H^2+g^{1\nu}\partial_2H^1\right\}\\&-g^{\nu 2}\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}-g^{\nu\hat{\alpha}}\frac{\partial\Phi^2_a}{\partial A^{\hat{\alpha}}_n}+g^{11}\frac{\partial\Phi^\nu_a}{\partial A^2_n}-g^{\nu 1}\frac{\partial\Phi^1_a}{\partial A^2_n}=0.\end{aligned} \quad (36)$$

Coefficient of $\partial_{\hat{\alpha}}\partial_3A^{\hat{\alpha}}_n$, with $\hat{\alpha}=0,1,2$:

$$\begin{aligned}&\delta_{an}\left\{-g^\nu_{\hat{\alpha}}(\partial^{\hat{\alpha}}H^3+2\partial^3H^{\hat{\alpha}})+g^{3\nu}(\partial_{\hat{\alpha}}H^{\hat{\alpha}}+2\partial^1H^1)+g^{\hat{\alpha}}_{\hat{\alpha}}\partial^\nu H^3+g^{1\nu}\partial_3H^1\right\}\\&-g^{\nu 3}\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}-g^{\nu\hat{\alpha}}\frac{\partial\Phi^3_a}{\partial A^{\hat{\alpha}}_n}+g^{11}\frac{\partial\Phi^\nu_a}{\partial A^3_n}-g^{\nu 1}\frac{\partial\Phi^1_a}{\partial A^3_n}=0.\end{aligned} \quad (37)$$

We can now investigate the conditions under which Eqs. (30)–(37) vanish. Let us first consider the case where $a\neq n$. Making use of Eq. (25), it is easy to see that (30)–(33) hold identically, whereas (34)–(37) hold if and only if, $\forall\hat{\alpha}$ and $\forall a,n\neq$

$$\frac{\partial\Phi^1_a}{\partial A^1_n}=\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}. \quad (38)$$

Let us now turn to the case where $a=n$. Substituting (28) into (30)–(33), we see that the latter vanish if and only if, $\forall\hat{\alpha}$

$$\partial_1H^1=\partial_{\hat{\alpha}}H^{\hat{\alpha}}. \quad (39)$$

Substituting Eqs. (27), (28) and (39) into (34)–(37), we see that the latter vanish if and only if, $\forall\hat{a},\hat{\alpha}$

$$\frac{\partial\Phi^1_{\hat{a}}}{\partial A^1_{\hat{a}}}=\frac{\partial\Phi^{\hat{\alpha}}_{\hat{a}}}{\partial A^{\hat{\alpha}}_{\hat{a}}}. \quad (40)$$

Eqs. (38) and (40) can be combined in the following, which holds $\forall a,n,\hat{\alpha}$:

$$\frac{\partial\Phi^1_a}{\partial A^1_n}=\frac{\partial\Phi^{\hat{\alpha}}_a}{\partial A^{\hat{\alpha}}_n}. \quad (41)$$

$\partial A \partial A \partial A$ terms

Since no such terms appear in the Yang-Mills equations, the substitution effected before Eq. (30) will not change the coefficients of $\partial A \partial A \partial A$ terms in (20). Owing to Eq. (23), these coefficients identically vanish.

$\partial A \partial A$ terms

Again, no such terms appear in the Yang-Mills equations. So we substitute Eq. (23) in the coefficients of $\partial A \partial A$ terms, symmetrize over the interchange of $(\lambda p \beta)$ with $(\kappa n \alpha)$ and set the result to zero. This yields

$$2g^{\kappa\lambda}\frac{\partial^2 \Phi_a^\nu}{\partial A_p^\beta \partial A_n^\alpha} - g^{\nu\lambda}\frac{\partial^2 \Phi_a^\kappa}{\partial A_p^\beta \partial A_n^\alpha} - g^{\nu\kappa}\frac{\partial^2 \Phi_a^\lambda}{\partial A_p^\beta \partial A_n^\alpha} = 0. \tag{42}$$

Eq. (42) holds $\forall \kappa, \lambda, \beta, \alpha, \nu, p, n, a$. So we must have

$$\frac{\partial^2 \Phi_a^\nu}{\partial A_p^\beta \partial A_n^\alpha} = 0. \tag{43}$$

That is, all second-order derivatives of Φ with respect to A vanish.

∂A terms

Here the situation is more complicated. There are such terms in the Yang-Mills equations. Therefore, the substitution effected before Eq. (30) does change the coefficients of ∂A terms in (20). We recall that we eliminated the following derivatives: $\partial_1\partial_1 A_n^0$, $\partial_2\partial_2 A_n^1$, $\partial_1\partial_1 A_n^2$, and $\partial_1\partial_1 A_n^3$. Taking (25) and (28) into account, we can see that for $\hat{\mu} \neq \alpha$ and $\hat{\mu} \neq 0$, the coefficient of $\partial_{\hat{\mu}}\partial_{\hat{\mu}} A_n^\alpha$ in Eq. (20) is given by

$$-2\delta_{an} g_\alpha^\nu \partial^{\hat{\mu}} H^{\hat{\mu}} - \frac{\partial \Phi_a^\nu}{\partial A_n^\alpha} = K_{an\alpha}^{\hat{\mu}\nu}. \tag{44}$$

Let $K_{an\alpha}^{\hat{\mu}\nu}$ denote the left-hand side of (44) for any value of the indices. Then ∂A terms coming from substitution of Eqs. (13)–(16) can be written as

$$K_{ac\kappa}^{1\nu} C_{cbn} \left\{2A_b^\alpha \partial_\alpha A_n^\kappa - A_b^\kappa \partial_\alpha A_n^\alpha - A_{b\alpha}\partial^\kappa A_n^\alpha\right\}, \tag{45}$$

where we have used the fact that, owing to Eq. (39), $K_{ac1}^{2\nu} = K_{ac1}^{1\nu}$. The previous expression can be rearranged as

$$(\partial_\lambda A_n^\alpha) A_b^\mu K_{ac\kappa}^{1\nu} C_{cbn} (2g_\mu^\lambda g_\alpha^\kappa - g_\alpha^\lambda g_\mu^\kappa - g^{\lambda\kappa} g_{\mu\alpha}). \tag{46}$$

The complete set of ∂A terms can now be obtained by adding the explicit ones in Eq. (20) to expression (46). Setting their coefficients equal to zero and rearranging, we find that $\forall \lambda, \nu, \alpha, n, a$

$$\begin{aligned}
&C_{adn}\left\{2g_\alpha^\nu \Phi_d^\lambda - g^{\nu\lambda}\Phi_{d\alpha} - g_\alpha^\lambda \Phi_d^\nu\right\} + \delta_{an}\left\{-g_\alpha^\nu \partial_\mu \partial^\mu H^\lambda + \partial^\nu \partial_\alpha H^\lambda\right\} \\
&\quad + 2\frac{\partial}{\partial A_n^\alpha}\partial^\lambda \Phi_a^\nu - \frac{\partial}{\partial A_n^\alpha}\partial^\nu \Phi_a^\lambda - g^{\nu\lambda}\frac{\partial}{\partial A_n^\alpha}\partial_\mu \Phi_a^\mu \\
&\quad + A_b^\mu \Big\{C_{abn}\left[-2g_\alpha^\nu \partial_\mu H^\lambda + g_\mu^\nu \partial_\alpha H^\lambda + g_{\alpha\mu}\partial^\nu H^\lambda - 2g_\kappa^\nu \partial^1 H^1 (2g_\mu^\lambda g_\alpha^\kappa - g_\alpha^\lambda g_\mu^\kappa - g_{\mu\alpha} g^{\lambda\kappa})\right] \\
&\quad + C_{abd}\left[2g_\mu^\lambda \frac{\partial \Phi_d^\nu}{\partial A_n^\alpha} - g_\mu^\nu \frac{\partial \Phi_d^\lambda}{\partial A_n^\alpha} - g^{\nu\lambda}\frac{\partial \Phi_{d\mu}}{\partial A_n^\alpha}\right] \\
&\quad - C_{cbn}\left[\frac{\partial \Phi_a^\nu}{\partial A_c^\kappa}\right](2g_\mu^\lambda g_\alpha^\kappa - g_\alpha^\lambda g_\mu^\kappa - g_{\mu\alpha} g^{\lambda\kappa})\Big\} = 0.
\end{aligned} \tag{47}$$

No-derivative terms

There are no-derivative terms in the Yang-Mills equations. Therefore, the substitution effected before Eq. (30) does change the coefficients of no-derivative terms in (20). Terms coming from the substitution are given by

$$K^{1\nu}_{an\kappa} C_{nbc} C_{clm} A^\mu_l A^\kappa_m A_{b\mu}. \tag{48}$$

The complete set of no-derivative terms can be obtained by adding the explicit ones in Eq. (20) to expression (48). Setting their coefficients equal to zero and making use of Eq. (44), we find that $\forall \nu, a$

$$\begin{aligned}
&\partial_\lambda \partial^\lambda \Phi^\nu_a - \partial_\lambda \partial^\nu \Phi^\lambda_a + A^\lambda_b C_{abd} \left\{2\partial_\lambda \Phi^\nu_d - g^\nu_\lambda \partial_\kappa \Phi^\kappa_d - \partial^\nu \Phi_{d\lambda}\right\} \\
&\quad + A^\mu_l A_{b\kappa} \left\{ g^\nu_\mu (C_{abc} C_{cdl} + C_{adc} C_{cbl}) \Phi^\kappa_d + g^\kappa_\mu C_{abc} C_{cld} \Phi^\nu_d \right\} \\
&\quad - A^\mu_l A^\kappa_m A_{b\mu} C_{nbc} C_{clm} \left\{ 2\delta_{an} g^\nu_\kappa \partial^1 H^1 + \frac{\partial \Phi^\nu_a}{\partial A^\kappa_n} \right\} = 0.
\end{aligned} \tag{49}$$

We have now obtained all determining equations associated with the Yang-Mills equations. They are given by Eqs. (23), (25), (27), (28), (39), (41), (43), (47) and (49). They are necessary and sufficient conditions for Eq. (20) to hold whenever the Yang-Mills equations hold.

4 Solution of Determining Equations

We now proceed to solve the determining equations. We first note that the most general solution of Eqs. (23) and (43) is given by

$$H^\mu = H^\mu(x^\lambda) \tag{50}$$

and

$$\Phi^\mu_a = \bar{f}^\mu_{ab\kappa}(x^\lambda) A^\kappa_b + F^\mu_a(x^\lambda), \tag{51}$$

where H^μ, $\bar{f}^\mu_{ab\kappa}$ and F^μ_a are arbitrary functions of x^λ. From Eq. (25), we see that $\bar{f}^\mu_{ab\kappa} = 0$ if $a \neq b$ and $\mu \neq \kappa$. We can therefore write

$$\Phi^{\hat\mu}_{\hat a} = \sum_{\kappa \neq \hat\mu} \bar{f}^{\hat\mu}_{\hat a \hat a \kappa}(x^\lambda) A^\kappa_{\hat a} + \bar{f}^{\hat\mu}_{\hat a b \hat\mu}(x^\lambda) A^{\hat\mu}_b + F^{\hat\mu}_{\hat a}(x^\lambda), \tag{52}$$

From Eq. (28) we see that, $\forall \hat a$ and $\forall \mu, \kappa \neq$

$$\partial_\kappa H^\mu = \bar{f}^\mu_{\hat a \hat a \kappa}. \tag{53}$$

Thus we can write, $\forall \hat a$ and $\forall \mu, \kappa \neq$

$$\bar{f}^\mu_{\hat a \hat a \kappa} = f^\mu{}_\kappa, \tag{54}$$

where, owing to Eq. (27), $f^{\mu\kappa}$ is antisymmetric.

From Eq. (39), we see that $\partial_{\hat\alpha} H^{\hat\alpha}$ is independent of $\hat\alpha$, and can therefore be written as G. From (41), we see that $\bar{f}^{\hat\mu}_{ab\hat\mu}$ is independent of $\hat\mu$, and can therefore be written as h_{ab}. The upshot is that the most general solution of Eqs. (23), (25), (27), (28), (39), (41) and (43) can be written as

$$\partial_\kappa H^\mu = f^\mu{}_\kappa + g^\mu_\kappa G \tag{55}$$

and

$$\Phi^\mu_a = f^\mu{}_\kappa A^\kappa_a + h_{ab} A^\mu_b + F^\mu_a, \tag{56}$$

where $f^{\mu\kappa} = -f^{\kappa\mu}$, h_{ab}, F^μ_a and G are arbitrary functions of x^λ. Note that

$$\frac{\partial \Phi^\mu_a}{\partial A^\kappa_b} = g^\mu_\kappa h_{ab} + \delta_{ab} f^\mu{}_\kappa. \tag{57}$$

There remains to satisfy Eqs. (47) and (49). We first substitute Eqs. (55)–(57) into (47). After cancellations and rearrangement, we find that $\forall \lambda, \nu, \alpha, n, a$

$$\begin{aligned} A^\mu_b &\left\{2g^\nu_\alpha g^\lambda_\mu - g^\nu_\mu g^\lambda_\alpha - g_{\alpha\mu} g^{\lambda\nu}\right\} \{C_{abn} G + C_{abd} h_{dn} - C_{dbn} h_{ad} + C_{adn} h_{db}\} \\ &+ \delta_{an} \left\{g^\lambda_\alpha \partial^\nu G - g^\nu_\alpha \partial_\mu (f^{\lambda\mu} + g^{\mu\lambda} G) + 2\partial^\lambda f^\nu{}_\alpha - g^{\lambda\nu} \partial_\mu f^\mu{}_\alpha\right\} \\ &+ 2g^\nu_\alpha \left\{C_{adn} F^\lambda_d + \partial^\lambda h_{an}\right\} - g^{\nu\lambda} \{C_{adn} F_{d\alpha} + \partial_\alpha h_{an}\} - g^\lambda_\alpha \{C_{adn} F^\nu_d + \partial^\nu h_{an}\} = 0. \end{aligned} \tag{58}$$

Since $f^{\mu\kappa}$, h_{ab}, F^μ_a and G are functions of x^λ only, it is clear that the coefficient of A^μ_b and the sum of terms independent of A must separately vanish. By considering cases where $a = n$ and $a \neq n$, we find that necessary and sufficient conditions for this are the following: First, $\forall a, b, n$

$$C_{abn} G + C_{abd} h_{dn} - C_{dbn} h_{ad} + C_{adn} h_{db} = 0. \tag{59}$$

Furthermore, $\forall \lambda$ and $\forall a, n \neq$

$$C_{adn} F^\lambda_d + \partial^\lambda h_{an} = 0. \tag{60}$$

Finally, $\forall \lambda, \nu, \alpha, \hat{a}$

$$\begin{aligned} g^\lambda_\alpha (\partial^\nu G - \partial^\nu h_{\hat{a}\hat{a}}) &- g^\nu_\alpha (\partial_\mu f^{\lambda\mu} + \partial^\lambda G - 2\partial^\lambda h_{\hat{a}\hat{a}}) \\ &+ 2\partial^\lambda f^\nu{}_\alpha - g^{\lambda\nu} (\partial_\mu f^\mu{}_\alpha + \partial_\alpha h_{\hat{a}\hat{a}}) = 0. \end{aligned} \tag{61}$$

Note that Eq. (60) can be written in a form that holds $\forall \lambda, \hat{a}, n$:

$$C_{\hat{a}dn} F^\lambda_d = -\partial^\lambda h_{\hat{a}n} + \delta_{\hat{a}n} \partial^\lambda h_{\hat{a}\hat{a}}. \tag{62}$$

In Eq. (61), set $\nu = \alpha \neq \lambda$. There results, $\forall \lambda, \hat{a}$

$$\partial_\mu f^{\lambda\mu} + \partial^\lambda G - 2\partial^\lambda h_{\hat{a}\hat{a}} = 0. \tag{63}$$

Note that this implies that $\partial^\lambda h_{\hat{a}\hat{a}}$ is independent of $\hat{a}$. Substituting (63) back into (61) yields $\forall \lambda, \nu, \alpha, \hat{a}$

$$g^\lambda_\alpha (\partial^\nu G - \partial^\nu h_{\hat{a}\hat{a}}) + 2\partial^\lambda f^\nu{}_\alpha - g^{\lambda\nu} (\partial_\alpha G - \partial_\alpha h_{\hat{a}\hat{a}}) = 0. \tag{64}$$

For $\nu = \alpha$, this vanishes identically. If $\nu \neq \alpha$, we can have any of three mutually exclusive cases: (i) $\alpha = \lambda \neq \nu$; (ii) $\nu = \lambda \neq \alpha$; (iii) $\nu, \lambda, \alpha \neq$. Case (i) yields $\forall \hat{a}$ and $\forall \nu, \hat{\alpha} \neq$

$$\partial^\nu G - \partial^\nu h_{\hat{a}\hat{a}} + 2\partial^{\hat{\alpha}} f^\nu{}_{\hat{\alpha}} = 0. \tag{65}$$

Case (ii) yields a similar equation. Finally, case (iii) yields, $\forall \lambda, \nu, \alpha \neq$

$$\partial^\lambda f^\nu{}_\alpha = 0. \tag{66}$$

Eqs. (59), (60), (63), (65) and (66) represent all the conditions on the unknown functions $f^{\mu\kappa}$, h_{ab}, F^μ_a and G provided by Eq. (47).

We now substitute Eqs. (55)–(57) in Eq. (49). After rearrangement, we find that $\forall \nu, a$

$$\begin{aligned}
&A_l^\mu A_m^\kappa A_b^\alpha \{g_{\alpha\mu} g_\kappa^\nu [C_{abc}C_{cld}h_{dm} + C_{nbc}C_{clm}(2\delta_{an}G - h_{an})] \\
&\quad + g_\mu^\nu (C_{abc}C_{cml} + C_{amc}C_{cbl}) f_{\alpha\kappa} + g_{\alpha\kappa} g_\mu^\nu (C_{abc}C_{cdl} + C_{adc}C_{cbl}) h_{dm}\Big\} \\
&\quad + A_l^\mu A_b^\alpha \Big\{g_\mu^\nu [(C_{abc}C_{cdl} + C_{adc}C_{cbl}) F_{d\alpha} - C_{alb}\partial_\kappa f^\kappa{}_\alpha - C_{ald}\partial_\alpha h_{db}] \\
&\quad + g_{\alpha\mu} [C_{abc}C_{cld}F_d^\nu - C_{ald}\partial^\nu h_{db}] + 2g_\alpha^\nu C_{ald}\partial_\mu h_{db} + C_{alb}(2\partial_\mu f^\nu{}_\alpha - \partial^\nu f_{\mu\alpha})\} \\
&\quad + A_l^\mu \Big\{C_{ald}(2\partial_\mu F_d^\nu - g_\mu^\nu \partial_\alpha F_d^\alpha - \partial^\nu F_{d\mu}) + \delta_{al}(\partial_\lambda \partial^\lambda f^\nu{}_\mu - \partial_\lambda \partial^\nu f^\lambda{}_\mu) + g_\mu^\nu \partial_\lambda \partial^\lambda h_{al} - \partial_\mu \partial^\nu h_{al}\Big\} \\
&\quad + \partial_\lambda \partial^\lambda F_a^\nu - \partial_\lambda \partial^\nu F_a^\lambda = 0. \qquad (67)
\end{aligned}$$

The (appropriately symmetrized) coefficients of each power of A must separately vanish. Let us consider each of them in turn.

It is not difficult to see that, owing to Eq. (60), terms independent of A identically vanish. Terms linear in A yield, $\forall \nu, \mu, a, l$

$$C_{ald}(2\partial_\mu F_d^\nu - g_\mu^\nu \partial_\alpha F_d^\alpha - \partial^\nu F_{d\mu}) + \delta_{al}(\partial_\lambda \partial^\lambda f^\nu{}_\mu - \partial_\lambda \partial^\nu f^\lambda{}_\mu) + g_\mu^\nu \partial_\lambda \partial^\lambda h_{al} - \partial_\mu \partial^\nu h_{al} = 0. \qquad (68)$$

For $a \neq l$, Eq. (60) implies that this holds identically. For $a = l$, we have $\forall \nu, \mu, \hat{a}$

$$\partial_\lambda \partial^\lambda f^\nu{}_\mu - \partial_\lambda \partial^\nu f^\lambda{}_\mu + g_\mu^\nu \partial_\lambda \partial^\lambda h_{\hat{a}\hat{a}} - \partial_\mu \partial^\nu h_{\hat{a}\hat{a}} = 0. \qquad (69)$$

Setting $\mu = \nu$ and summing immediately yields $\forall \hat{a}$

$$\partial_\lambda \partial^\lambda h_{\hat{a}\hat{a}} = 0, \qquad (70)$$

whence $\forall \nu, \mu, \hat{a}$

$$\partial_\lambda \partial^\lambda f^\nu{}_\mu - \partial_\lambda \partial^\nu f^\lambda{}_\mu - \partial_\mu \partial^\nu h_{\hat{a}\hat{a}} = 0. \qquad (71)$$

We turn to terms quadratic in A in Eq. (67). The coefficient of these terms, symmetrized under the interchange $(\mu, l) \leftrightarrow (\alpha, b)$, must vanish. A rather lengthy but straightforward calculation, which we shall not reproduce here, shows that, owing to (59), (63), (66) and the antisymmetry of $f_{\mu\alpha}$, the resulting equation reduces to an identity. Similarly, the coefficient of terms cubic in A, symmetrized under the sixfold interchange $(\mu, l) \leftrightarrow (\alpha, b) \leftrightarrow (\kappa, m)$, vanishes identically. The upshot is that Eqs. (70) and (71) represent all additional conditions on the unknown functions $f^{\mu\kappa}$, h_{ab}, F_a^μ and G provided by Eq. (49).

We now proceed to solve Eqs. (59), (60), (63), (65), (66), (70) and (71). First, let us write (63), (65), (70) and (71) in a simpler form. Consider Eq. (65) for the three values of $\hat{\alpha} \neq \nu$. Summing the three resulting equations and remembering that $f^\nu{}_\alpha$ vanishes if $\nu = \alpha$, we get $\forall \nu, \hat{a}$

$$3\partial^\nu G - 3\partial^\nu h_{\hat{a}\hat{a}} + 2\partial^\alpha f^\nu{}_\alpha = 0. \qquad (72)$$

Comparing with (63), we find that $\forall \nu, \hat{a}$

$$\partial^\nu G + \partial^\nu h_{\hat{a}\hat{a}} = 0. \qquad (73)$$

Substituting (73) into (63), (71) and (65) and relabelling yields, $\forall \mu, \nu$

$$\partial_\lambda f^{\mu\lambda} + 3\partial^\mu G = 0, \qquad (74)$$

$$\partial_\lambda \partial^\lambda f^{\nu\mu} - \partial_\lambda \partial^\nu f^{\lambda\mu} + \partial^\nu \partial^\mu G = 0, \qquad (75)$$

and, $\forall\mu, \hat{\lambda} \neq$

$$\partial_{\hat{\lambda}} f^{\mu\hat{\lambda}} + \partial^\mu G = 0. \tag{76}$$

Substituting (74) in (75) yields $\forall\mu, \nu$

$$\partial_\lambda \partial^\lambda f^{\nu\mu} = 2\partial^\nu \partial^\mu G. \tag{77}$$

Since one side is antisymmetric under the interchange $\nu \leftrightarrow \mu$ and the other side is symmetric, both sides must vanish. So we have, $\forall\nu, \mu$

$$\partial^\nu \partial^\mu G = 0. \tag{78}$$

Owing to (66), Eqs. (73), (76) and (78) are equivalent to (63), (65), (70) and (71).

The most general solution of Eq. (78) is given by

$$G = d + c_\mu x^\mu, \tag{79}$$

where d and c_μ are arbitrary constants. From (66), we see that $f^{\nu\alpha}$ is a function of x^ν and x^α only. From (76) and (79) we obtain $\forall\mu, \hat{\lambda} \neq$

$$\partial_{\hat{\lambda}} f^{\mu\hat{\lambda}} = -\partial^\mu G = -c^\mu. \tag{80}$$

This implies that the most general solution for $f^{\mu\lambda}$ is

$$f^{\mu\lambda} = -c^\mu x^\lambda + c^\lambda x^\mu + b^{\mu\lambda}, \tag{81}$$

where $b^{\mu\lambda}$ are six arbitrary constants such that $b^{\mu\lambda} = -b^{\lambda\mu}$.

We can now solve for the functions $H^\mu(x^\lambda)$. With (79) and (81), Eq. (55) can easily be integrated to give

$$H^\mu = -\frac{1}{2} c^\mu x^\lambda x_\lambda + c^\lambda x^\mu x_\lambda + b^{\mu\lambda} x_\lambda + d x^\mu + a^\mu, \tag{82}$$

where a^μ are four arbitrary constants.

There remains to solve Eqs. (59), (60) and (73). In Appendix B, we shall show by group theoretical arguments that the most general solution of Eq. (59) is given by

$$h_{ab} = -G\delta_{ab} + C_{abd}\chi_d, \tag{83}$$

where the χ_d are arbitrary functions of x^λ. Substituting Eq. (83) in (73), we see that the latter holds identically. Substituting (83) in (60), we find that $\forall\lambda$ and $\forall a, n \neq$

$$C_{adn} F^\lambda_d = -\partial^\lambda C_{and}\chi_d, \tag{84}$$

whence, owing to Eq. (5) and the antisymmetry of the structure constants

$$F^\lambda_d = \partial^\lambda \chi_d. \tag{85}$$

Putting together Eqs. (56), (79), (81), (83) and (85), we find that

$$\Phi^\mu_a = (-c^\mu x_\lambda + c_\lambda x^\mu + b^\mu{}_\lambda) A^\lambda_a - (d + c_\lambda x^\lambda) A^\mu_a + C_{abd}\chi_d A^\mu_b + \partial^\mu \chi_a. \tag{86}$$

Eqs. (82) and (86) are the most general solution of the determining equations. Therefore, the corresponding vector field (7) generates Lie symmetries of the Yang-Mills equations. One can see that the constants a^μ correspond to space-time translations; that the $b^{\lambda\mu}$ correspond to Lorentz transformations; that the c^μ correspond to uniform accelerations; that d corresponds to dilatations; and that the functions $\chi_a(x^\lambda)$ correspond to local gauge transformations [8, 9]. We have thus recovered the well-known Lie symmetries of the Yang-Mills equations. But we have done much more. Isofar as the Yang-Mills equations are locally solvable, we have shown that there are no others.

5 Gauge Conditions

In Eqs. (82) and (86), we have obtained the coefficients of symmetry generators of the Yang-Mills equations. In practice, the equations will be used together with a gauge condition. So it is of interest to investigate the symmetries of the Yang-Mills equations in a particular gauge. To be specific, we shall pick the Lorentz gauge.

The Lorentz gauge condition consists in setting $\forall a$

$$\partial_\mu A_a^\mu = 0. \tag{87}$$

Our task consists in finding the Lie symmetries of Eqs. (6) and (87).

It is not difficult to check that Eqs. (6) and (87) together have maximal rank. But they are not locally solvable. Differentiating (87) with respect to x^λ, we find that

$$\partial_\lambda \partial_\mu A_a^\mu = 0, \tag{88}$$

which are additional constraints on partial derivatives.

It is shown in Ref. [1] that a necessary and sufficient condition for v to generate a symmetry of a system of n-th order equations is that the n-th prolongation of v, acting on the system, vanishes at all points where the system is locally solvable. In our case, such points are determined by Eqs. (6), (87), (88), and any other equation expressing constraints on the A_a^μ and their first and second-order derivatives. For similar reasons as given in Section 2, however, it is likely that there are no additional constraints. We shall thus investigate the conditions under which the second prolongation of v, acting on Eqs. (6) and (87), vanish whenever Eqs. (6), (87) and (88) hold.

Let us apply the second prolongation operator (8) to Eqs. (6) and (87), and set the result to zero. Applying (8) to Eq. (6), we clearly recover Eq. (20). Applying (8) to (87), we find that

$$\Phi_{a\mu}^\mu = 0, \tag{89}$$

or, using (17)

$$\partial_\mu \Phi_a^\mu - (\partial_\mu H^\nu)\partial_\nu A_a^\mu + (\partial_\mu A_n^\alpha)\frac{\partial}{\partial A_n^\alpha}\Phi_a^\mu - (\partial_\mu A_n^\alpha)(\partial_\nu A_a^\mu)\frac{\partial}{\partial A_n^\alpha}H^\nu = 0. \tag{90}$$

We now have to substitute Eqs. (6), (87), and (88) into (20) and (90), and equate to zero the coefficients of the remaining (independent) combinations of derivatives of A. Note that this complicated and correct procedure is not the same as the simpler one that consists in substituting Eqs. (82) and (86) into (90), although in specific instances the two procedures may yield the same results.

Let us then consider in turn the various combinations of derivatives of A. The $\partial A \partial\partial A$ terms can be treated basically as in Section 2. We recall that only terms $\partial_\lambda \partial_\mu A_p^\beta$, with λ, μ and β all different, had to be considered. Thus, substitution of (88) will not have any effect. Moreover, it is not difficult to see that Eq. (23) still obtains if we restrict our attention to terms $\partial^\kappa A_n^\alpha$ with $\kappa \neq \alpha$. But the Lorentz gauge condition does not involve such terms. Eqs. (23), therefore, are still necessary and sufficient conditions for the $\partial A \partial\partial A$ terms to vanish.

Discussion of $\partial\partial A$ terms is not much changed either. Eq. (88) allows to write terms like $\partial_{\hat\alpha}\partial_{\hat\alpha} A_a^{\hat\alpha}$ in terms of other second-order derivatives of A. But the coefficient of $\partial_{\hat\alpha}\partial_{\hat\alpha} A_a^{\hat\alpha}$ is given by the left-hand side of Eq. (29), which was shown to vanish identically. So again, the substitution of the Lorentz gauge condition and its derivatives will not introduce anything new.

It is easy to see that $\partial A \partial A \partial A$ terms still vanish identically. Turning to $\partial A \partial A$ terms, we can see that Eq. (43) can be obtained even if we restrict our attention to terms with $\lambda \neq \beta$ and $\kappa \neq \alpha$.

The ∂A terms yield Eqs. (59), (60), (63), (65) and (66) even if we restrict ourselves to $\lambda \neq \alpha$. Finally, terms with no derivatives of A do not change, since the Lorentz gauge condition involves derivatives only.

The upshot of the foregoing analysis is that the conditions that make (20) vanish subject to (6), (87), and (88) are the same as the ones that make (20) vanish subject to (6) only. In the end, these conditions are precisely embodied in Eqs. (82) and (86). We stress that this is not obvious, and could be otherwise for other choices of gauge.

There remains to make use of Eq. (90) to put further constraints on the functions H^μ and Φ^μ_a. Substituting (82) and (86) into (90) and rearranging, we find that

$$2c_\kappa A^\kappa_a + C_{abd}(\partial_\mu \chi_d) A^\mu_b + \partial_\mu \partial^\mu \chi_a = 0. \tag{91}$$

This must hold identically. Since χ_a is a function of x^λ only, we get

$$\partial_\mu \partial^\mu \chi_a = 0, \tag{92}$$

$$2c_\mu \delta_{ab} + C_{abd} \partial_\mu \chi_d = 0. \tag{93}$$

Necessary and sufficient conditions for these two equations to hold are that $(\forall \mu)$, $c_\mu = 0$ and that $(\forall \mu, d)$, $\partial_\mu \chi_d = 0$. The conformal symmetry thus collapses to the Poincaré group with dilatations, and local gauge transformations reduce to global ones.

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A

To see whether there are additional constraints on the A^ν_a and their derivatives, let us apply the operator ∂_ν on Eq. (6). We get

$$\begin{aligned} C_{abc} \{ & 2A^\mu_b \partial_\nu \partial_\mu A^\nu_c + 2(\partial_\nu A^\mu_b)\partial_\mu A^\nu_c + (\partial_\nu \partial_\mu A^\mu_b) A^\nu_c + (\partial_\mu A^\mu_b)\partial_\nu A^\nu_c - (\partial_\nu A_{b\mu})\partial^\nu A^\mu_c \\ & - A_{b\mu}\partial_\nu \partial^\nu A^\mu_c + C_{cdl}\left[A_{b\mu} A^\nu_l \partial_\nu A^\mu_d + A_{b\mu} A^\mu_d \partial_\nu A^\nu_l + A^\mu_d A^\nu_l \partial_\nu A_{b\mu}\right] \} = 0. \end{aligned} \tag{94}$$

The second, fourth, and fifth terms in curly brackets vanish due to antisymmetry of C_{abc}. The third term similarly cancels half the first term. Thus we obtain

$$C_{abc} \left\{ A^\mu_b \partial_\nu \partial_\mu A^\nu_c - A_{b\mu}\partial_\nu \partial^\nu A^\mu_c + C_{cdl}\left[A_{b\mu} A^\nu_l \partial_\nu A^\mu_d + A_{b\mu} A^\mu_d \partial_\nu A^\nu_l + A^\mu_d A^\nu_l \partial_\nu A_{b\mu}\right]\right\} = 0. \tag{95}$$

Substituting Eq. (6) and again making use of the antisymmetry of the structure constants, we get

$$C_{abc}C_{cdl} \left\{ A_{b\mu} A^\nu_d \partial_\nu A^\mu_l - A_{b\mu} A_{d\nu} \partial^\mu A^\nu_l + A^\mu_d A^\nu_l \partial_\nu A_{b\mu} + C_{lmn} A_{b\mu} A^\nu_m A^\mu_n A_{d\nu} \right\} = 0. \tag{96}$$

Relabeling indices in the second and last terms and regrouping yields

$$\begin{aligned} & \{C_{abc}C_{cdl} + C_{adc}C_{clb}\} A_{b\mu} A^\nu_d \partial_\nu A^\mu_l + C_{abc}C_{cdl} A^\mu_d A^\nu_l \partial_\nu A_{b\mu} \\ & + \frac{1}{2}\{C_{abc}C_{cdl} + C_{adc}C_{clb}\} C_{lmn} A_{b\mu} A^\nu_m A^\mu_n A_{d\nu} = 0. \end{aligned} \tag{97}$$

Making use of the Jacobi identities for the structure constants, we get

$$-C_{alc}C_{cbd} A_{b\mu} A^\nu_d \partial_\nu A^\mu_l + C_{abc}C_{cdl} A^\mu_d A^\nu_l \partial_\nu A_{b\mu} - \frac{1}{2} C_{alc}C_{cbd}C_{lmn} A_{b\mu} A^\nu_m A^\mu_n A_{d\nu} = 0. \tag{98}$$

The first two terms cancel and, by antisymmetry of the structure constants, the third term vanishes.

B

We want to solve Eq. (59), namely

$$C_{abn}G + C_{abd}h_{dn} - C_{dbn}h_{ad} + C_{adn}h_{db} = 0. \tag{99}$$

The C_{abn} are structure constants of a compact semisimple Lie group, the function G is given by Eq. (79) and the h_{ad} are unknown functions of x^λ.

We fix the value of x^λ, so that G and h_{ad} are fixed too. In (99), we interchange a with n, and add the result to (99). We obtain

$$C_{abd}(h_{dn} + h_{nd}) + C_{nbd}(h_{da} + h_{ad}) = 0 \tag{100}$$

or, in matrix notation

$$\left[C_b, h + h^T\right] = 0, \tag{101}$$

where C_b has elements C_{bad}, h has elements h_{ad} and h^T is the transpose of h. Now the structure constants are matrices of an irreducible representation of the Lie algebra. By Schur's lemma, Eq. (101) implies that $h + h^T$ is a multiple of the identity, that is,

$$h_{ad} + h_{da} = 2\lambda\delta_{ad}. \tag{102}$$

Let us denote by M_{ad} the antisymmetric part of h_{ad}. Owing to (102), Eq. (99) becomes

$$C_{abn}G + C_{abd}(\lambda\delta_{dn} + M_{dn}) - C_{dbn}(\lambda\delta_{ad} + M_{ad}) + C_{adn}(\lambda\delta_{db} + M_{db}) = 0, \tag{103}$$

which reduces to

$$C_{abn}(G + \lambda) + C_{abd}M_{dn} + C_{bnd}M_{da} + C_{nad}M_{db} = 0. \tag{104}$$

We multiply this equation by C_{abl}, sum over a and b and make use of Eq. (5) to obtain

$$\delta_{ln}(G + \lambda) + M_{ln} + 2C_{abl}C_{bnd}M_{da} = 0. \tag{105}$$

The first term is symmetric under the interchange $l \leftrightarrow n$, whereas the last two terms are antisymmetric. This means that

$$G + \lambda = 0, \tag{106}$$

whence

$$h_{ad} = -G\delta_{ad} + M_{ad}. \tag{107}$$

Eq. (104) becomes

$$C_{abd}M_{dn} + C_{bnd}M_{da} + C_{nad}M_{db} = 0. \tag{108}$$

It is obvious that, for any set of χ_l, the following is a solution of Eq. (108):

$$M_{dn} = C_{dnl}\chi_l. \tag{109}$$

We shall now show that there are no other solutions.

Owing to (106), Eq. (105) can be written in matrix form as

$$M + 2\sum_b C_b M C_b = 0. \tag{110}$$

Let L denote the Lie algebra whose structure constants are the C_{bad}. Then L is semisimple. In a suitable basis, each matrix C_b is block diagonal, with nonzero entries in one block only. Each block

corresponds to a simple subalgebra of L. From Eq. (110), it follows that M is also block diagonal. Eq. (110), therefore, holds separately for each block. Thus it is enough to consider the case where L is simple.

From (108), we have

$$[C_a, M] = \sum_d M_{ad} C_d. \tag{111}$$

Suppose there is a matrix M that satisfies (108) and is not a linear combination of the C_d. From (111), we see that the C_d and M together form a Lie algebra that includes L. Let N denote the dimension of L, and let $\mathrm{M}(L;R)$ denote the real irreducible representation of L made up of the structure constants. Let $\mathrm{M}(L;C)$ denote the corresponding representation of the complex form of L. It is known that $\mathrm{M}(L;C)$ is maximal in the orthogonal algebra $\mathrm{SO}(N;C)$ [10]. From this it follows that $\mathrm{M}(L;R)$ is maximal in $\mathrm{SO}(N;R)$. For, if there existed a real Lie algebra L' such that

$$\mathrm{M}(L;R) \subset L' \subset \mathrm{SO}(N;R), \tag{112}$$

corresponding inclusions would also hold for the complex forms. But

$$\dim\{\mathrm{SO}(N;R)\} - \dim\{L\} = \frac{N(N-3)}{2}. \tag{113}$$

Since this is never equal to 1, the C_d and M cannot together form a Lie algebra. The upshot is that Eq. (109) is the most general solution of (108). Eq. (107) thus becomes

$$h_{ad} = -G\delta_{ad} + C_{abd}\chi_d. \tag{114}$$

Since this holds at any point x^λ, Eq. (83) follows.

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